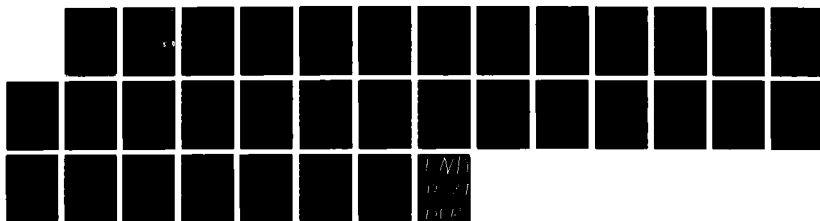
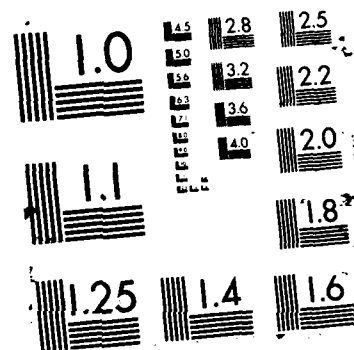


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Abstract

We present autoregressive (AR) and autoregressive moving average (ARMA) processes with bivariate exponential (BE) and bivariate geometric (BG) distributions. The theory of positive dependence is used to show that in various cases, the BEAR, BGAR, BEARMA, and BGARMA models consist of associated random variables. We discuss special cases of the BEAR and BGAR processes in which the bivariate processes are stationary and have well known bivariate exponential and geometric distributions.

Keywords and phrases: Bivariate exponential and geometric distributions, bivariate autoregressive and autoregressive moving average models in exponential and geometric random vectors, association, joint stationarity.

# 1. Introduction and Summary

A primary stationary model in time series analysis is the  $p \times 1$  linear process given by:

$$(1.1) \quad \underline{X}(n) = \sum_{j=-\infty}^{\infty} A(j)\underline{\epsilon}(n-j), \quad n = 0, \pm 1, \pm 2, \dots,$$

where  $A(j)$ ,  $j = 0, \pm 1, \pm 2, \dots$ , is a sequence of  $p \times p$  parameter matrices such that

$$\sum_{j=-\infty}^{\infty} ||A(j)|| < \infty, \text{ and } \underline{\epsilon}(n), \quad n = 0, \pm 1, \pm 2, \dots, \text{ is a sequence of uncorrelated}$$

$p \times 1$  random vectors with mean zero and common covariance matrix. It is well known that (1.1) includes the stationary vector autoregressive (AR) process and the stationary and invertible vector autoregressive moving average (ARMA) process. However, in some physical situations where the random vectors  $\underline{X}(n)$  are either positive or discrete, the preceding assumptions on the  $\underline{\epsilon}(n)$  sequence are inappropriate (see Lewis (1980), p. 152).

Several researchers, addressing themselves to this problem, have constructed univariate stationary AR type models and stationary ARMA type models where the random variables  $X(n)$  have exponential or gamma distributions, and discrete models where  $X(n)$  assumes values in a common set. Lawrence and Lewis (1977, 1980, 1981) and Jacobs and Lewis (1977) present stationary AR and ARMA type models where the random variables  $X(n)$  have exponential distributions; Gaver and Lewis (1980) consider stationary ARMA type models where the random variables  $X(n)$  have gamma distributions. Jacobs and Lewis (1978a,b, 1983) construct ARMA type models where the random variables  $X(n)$  are discrete and assume values in a common finite set. The aforementioned models have been used in the various fields of applied probability and time series analysis, for example, these models have been used to model and analyze univariate point processes with correlated service and correlated interarrival times (see

Jacobs (1978)). Details concerning bivariate exponential and geometric MA type processes and the corresponding point processes may be found in Langberg and Stoffer (1985).

In this paper we present two classes of AR and of ARMA type sequences of bivariate random vectors. The first class has exponential marginals while the second class has geometric marginals. We denote the first [second] class of models as BEAR( $m$ ) [BGAR( $m$ )] and BEARMA( $m_1, m_2$ ) [BGARMA( $m_1, m_2$ )] for bivariate exponential [geometric] autoregressive, order  $m$ , and autoregressive moving average, order  $(m_1, m_2)$ , respectively, where  $m$  and  $(m_1, m_2)$  parametrize the order of the dependence on the past. We use the theory of positive dependence to show that in a variety of cases the classes of sequences are associated.

In Section 2 we define the bivariate exponential and geometric distributions which are the underlying distributions of our two classes, and present a variety of examples of such distributions. Furthermore, in Section 2 we define the concept of association and present a variety of bivariate exponential and geometric distributions that are associated. We conclude Section 2 by describing the bivariate dependence mechanisms which are used in generating the various models. In Section 3 we construct the general BEAR( $m$ ) and BGAR( $m$ ) models showing that the sequences have bivariate exponential and bivariate geometric distributions, respectively. Also, we discuss the autocorrelation structure of the two classes of sequences. In Section 4 we consider special cases of the BEAR(1) and BGAR(1) sequences. We show that defined appropriately, the bivariate processes are stationary, and obtain well known bivariate exponential and geometric distributions. Finally, in Section 5, we present the BEARMA ( $m_1, m_2$ ) and BGARMA( $m_1, m_2$ ) models. We conclude by describing the association properties of the sequences and discussing how to

utilize association to obtain some probability bounds and moment inequalities for the bivariate processes and the corresponding point processes.

## 2. Preliminaries

In this section we present definitions and prove some basic results to be used in the sequel. First, we give definitions of bivariate geometric and exponential distributions and provide some examples. Then we present the concept of association and give some examples. Finally, we discuss some bivariate dependence mechanisms.

First, we present a definition of a bivariate geometric distribution.

Definition 2.1. Let  $M, N$  be random variables assuming values in the set  $\{1, 2, \dots\}$ . We say that  $(M, N)$  has a bivariate geometric distribution if  $M$  and  $N$  have geometric distributions.

Examples 2.2. (a) Let  $N$  be geometric. Then  $(N, N)$  is bivariate geometric. (b) Let  $M$  and  $N$  be independent geometric random variables, then  $(M, N)$  is bivariate geometric. (c) Let  $N_1, N_2, N_3$  be independent geometric random variables, and put  $M = \{\min(N_1, N_3)\}$ ,  $N = \{\min(N_2, N_3)\}$ . Then  $(M, N)$  has the Esary-Marshall (1974) bivariate geometric distribution. (d) Let  $p_{00}, p_{01}, p_{10}, p_{11}$  be in  $[0, 1]$  such that (i)  $p_{00} + p_{01} + p_{10} + p_{11} = 1$ , (ii)  $p_{01} + p_{11} < 1$  and  $p_{10} + p_{11} < 1$ , and let  $M, N$  be random variables assuming values in the set  $\{1, 2, \dots\}$  determined by:

$$(2.3) \quad P(M > m, N > n) = \begin{cases} p_{11}^m [p_{01} + p_{11}]^{n-m}, & n \geq m, \\ p_{11}^n [p_{10} + p_{11}]^{m-n}, & n \leq m, \quad m, n = 1, 2, \dots \end{cases}$$

Then  $(M,N)$  has the Block (1977) fundamental bivariate geometric distribution (see also Block and Paulson (1984)). (e) Let  $(M_1, M_2)$  be bivariate geometric and let  $(N_1(j), N_2(j))$ ,  $j = 1, 2, \dots$ , be an iid sequence of random vectors with bivariate geometric distributions which are independent of  $(M_1, M_2)$ . Then

$(\sum_{j=1}^{M_1} N_1(j), \sum_{j=1}^{M_2} N_2(j))$  has a bivariate geometric distribution (cf. Lemma 2.14).

In the following remark we show how some of the bivariate geometric distributions are particular cases of Example 2.2d. Other examples are given in Remark 4.10.

Remarks 2.4. (a) Let  $p_{10} = p_{01} = 0$  in equation (2.3). Then we obtain the distribution introduced in Example 2.2a. (b) Let  $p_{11} = (p_{11} + p_{10})(p_{11} + p_{01})$  in (2.3). Then we obtain the bivariate geometric distribution introduced in Example 2.2b. (c) Let  $p_{11} \geq (p_{11} + p_{10})(p_{11} + p_{01})$  in (2.3) and let  $N_1, N_2, N_3$  be independent geometric random variables with parameters  $p_{11}(p_{11} + p_{01})^{-1}$ ,  $p_{11}(p_{11} + p_{10})^{-1}$ ,  $p_{11}^{-1}(p_{11} + p_{10})(p_{11} + p_{01})$ , respectively. Put  $M = (\min(N_1, N_3))$  and  $N = (\min(N_2, N_3))$ . Then  $(M, N)$  is stochastically equal to the Esary-Marshall bivariate geometric distribution given in Example 2.2c.

Next, we present a definition of a bivariate exponential distribution.

Definition 2.5 Let  $E_1, E_2$  be random variables assuming values in  $(0, \infty)$ . We say that  $(E_1, E_2)$  has a bivariate exponential distribution if  $E_1$  and  $E_2$  have exponential distributions.

Examples 2.6 (a) Let  $E$  be exponential. Then  $(E, E)$  is bivariate exponential. (b) Let  $E_1, E_2$  be independent exponentials. Then  $(E_1, E_2)$  has a bivariate exponential distribution. (c) Let  $X_1, X_2, X_3$  be independent exponentials and put  $E_1 = (\min(X_1, X_3))$ ,  $E_2 = (\min(X_2, X_3))$ . Then  $(E_1, E_2)$  has the Marshall-Olkin

(1967) bivariate exponential distribution. (d) Let  $(M,N)$  have a bivariate geometric distribution and let  $(E_1(j), E_2(j))$ ,  $j = 1, 2, \dots$ , be an iid sequence of random vectors with bivariate exponential distributions, independent of  $M$  and  $N$ . Then  $(\sum_{j=1}^M E_1(j), \sum_{j=1}^N E_2(j))$  has a bivariate exponential distribution (cf. Lemma 2.14). (e) Let  $0 \leq \alpha \leq 1$ . Then  $(E_1, E_2)$  determined by  $P\{E_1 > x, E_2 > y\} = \exp\{-x-y-\alpha xy\}$ ,  $x, y > 0$ , has a Gumbel (1960) bivariate exponential distribution. (f) Let  $|\alpha| \leq 1$ . Then  $(E_1, E_2)$  determined by  $P\{E_1 \leq x, E_2 \leq y\} = (1-e^{-x})(1-e^{-y})(1+\alpha e^{-x-y})$ ,  $x, y > 0$ , has a bivariate Gumbel (1960) exponential distribution. (g) Let  $\alpha \geq 1$ . Then  $(E_1, E_2)$  determined by  $P\{E_1 > x, E_2 > y\} = e^{-(x^\alpha + y^\alpha)^{1/\alpha}}$ ,  $x, y > 0$ , is bivariate exponential. (h) Let  $(X, Y)$  be a random vector with continuous marginal distributions  $F$  and  $G$ , respectively. Then the random vector  $(-\ln[1-F(X)], -\ln[1-G(Y)])$  is bivariate exponential.

Example 2.6(d) has been used by several researchers to generate bivariate distributions (for example Arnold (1975), Downton (1970), and Hawkes (1972) to mention a few). In the following remarks we illustrate how some of the bivariate exponential distributions are obtained from Example 2.6(d). Other examples are given in Remark 4.5.

Remarks 2.7. (a)  $M = N$  and let  $E_1(j), E_2(j)$  be independent exponentials,  $j = 1, 2, \dots$ . Then we obtain the distribution introduced by Downton (1970). (b) Let  $(M, N)$  be as in Example 2.2(d) and let  $E_1(j), E_2(j)$  be independent exponentials,  $j = 1, 2, \dots$ . Then we obtain the bivariate exponential distribution introduced by Hawkes (1972) and Paulson (1973). (c) Let  $(M, N)$  be as in Example 2.2(c) and let  $E_1(j) = E_2(j)$ ,  $j = 1, 2, \dots$ . Then we obtain the Marshall-Olkin (1967) distribution given in Example 2.6(c) (for details see Marshall-Olkin (1967)).

Next, we present a concept of positive dependence.

Definition 2.8 Let  $\underline{T} = (T_1, \dots, T_n)$ ,  $n = 1, 2, \dots$ , be a multivariate random vector. We say that the random variables  $T_1, \dots, T_n$  are associated if  $\text{cov}(f(\underline{T}), g(\underline{T})) \geq 0$  for all  $f$  and  $g$  monotonically nondecreasing in each argument, such that the expectations exist.

Remarks 2.9. (a) Note that independent random variables are associated and that nondecreasing functions of associated random variables are associated (cf. Barlow and Proschan (1975) pp. 30-31). Thus the components of the vector given in Example 2.2(c) and the components of the vector given in Example 2.6(c) are associated. (b) Let  $(E_1, E_2)$  be as in Example 2.6(e), with  $\alpha > 0$ , or as in 2.6(f) with  $-1 \leq \alpha < 0$ . Since  $P(E_1 > x, E_2 > y) < P(E_1 > x)P(E_2 > y)$  for  $x, y > 0$ ,  $E_1$  and  $E_2$  are not associated. (c) Let  $(X, Y)$  be as in Example 2.6(h). Then  $-\ln[1-F(X)]$  and  $-\ln[1-G(Y)]$  are associated if and only if  $X$  and  $Y$  are associated (cf. Barlow and Proschan (1975), Proposition 3, p. 30). (d) The components of the bivariate geometric distribution given in Example 2.2(e) are associated provided  $M_1$  and  $M_2$ , and  $N_1(1)$  and  $N_2(1)$  are associated (cf. Langberg and Stoffer (1985), Lemma 2.10). (e) The components of the bivariate exponential distribution given in Example 2.6(d) are associated provided that  $M$  and  $N$ , and  $E_1(1)$  and  $E_2(1)$  are associated by the same reasoning as in Remark 2.9(d).

We are now ready to discuss the various dependence mechanisms used in obtaining bivariate exponential and geometric distributions. It turns out that many of these mechanisms are related and we describe these relationships.

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The notation  $X \stackrel{st}{=} Y$  will mean that  $X$  and  $Y$  are random variables (or vectors) with the same distribution.

Lemma 2.10. (Random Mixing) (a) Let  $(X_1, X_2)$  and  $(Z_1, Z_2)$  be independent random vectors with exponential marginals where  $X_1 \stackrel{st}{=} Z_1$ ,  $X_2 \stackrel{st}{=} Z_2$ , and  $(X_1, X_2)$ ,  $(Z_1, Z_2)$  have mean vector  $(\lambda_1^{-1}, \lambda_2^{-1})$ . Let  $(I_1, I_2)$  be a bivariate Bernoulli random vector independent of  $(X_1, X_2)$ ,  $(Z_1, Z_2)$ . Also, assume

$$(2.11) \quad P(I_1 = i, I_2 = j) = p_{ij} \quad i, j = 0, 1$$

such that  $\sum_{i,j} p_{ij} = 1$ ;  $1 - \pi_1 = p_{10} + p_{11} < 1$ , and  $1 - \pi_2 = p_{01} + p_{11} < 1$ . Then a random vector given by

$$(2.12) \quad (Y_1, Y_2) \stackrel{st}{=} (I_1 Z_1, I_2 Z_2) + (\pi_1 X_1, \pi_2 X_2)$$

has a bivariate exponential distribution with the same marginals as  $(X_1, X_2)$  and  $(Z_1, Z_2)$ . (b) Let  $(M_1, M_2)$  and  $(K_1, K_2)$  be independent geometric random vectors such that for  $k = 1, 2, \dots$ ,  $P(M_i = k) = (p_i/\pi_i)(1-p_i/\pi_i)^{k-1}$  and  $P(K_i = k) = p_i(1-p_i)^{k-1}$ ,  $i = 1, 2$ , with  $0 < p_1, p_2 < 1$ . Let  $(I_1, I_2)$  be defined by (2.11) and be independent of  $(M_1, M_2)$ ,  $(K_1, K_2)$ . Then a random vector given by

$$(2.13) \quad (G_1, G_2) \stackrel{st}{=} (I_1 K_1, I_2 K_2) + (M_1, M_2)$$

has the same marginals as  $(K_1, K_2)$ .

Proof: Parts (a) and (b) follow easily by computing the marginal characteristic functions. ||

Raftery (1985) gives a special case of Lemma 2.10a where  $Z_1 = Z_2$ .

Lemma 2.14 (Random Summation). Let  $(X_{1j}, X_{2j}) [(M_{1j}, M_{2j})]$ ,  $j = 1, 2, \dots$ , be iid bivariate exponential [geometric] random vectors with mean vector  $(\pi_1/\lambda_1, \pi_2/\lambda_2) [(\pi_1/p_1, \pi_2/p_2)]$ ,  $0 < \pi_1, \pi_2 < 1$ . Let  $(N_1, N_2)$  have a bivariate

geometric distribution with mean vector  $(\pi_1^{-1}, \pi_2^{-1})$  and be independent of the  $(X_{1j}, X_{2j})$  and the  $(M_{1j}, M_{2j})$ . Then random vectors given by

$$(2.15) \quad (Y_1, Y_2) \stackrel{st}{=} \left( \sum_{j=1}^{N_1} X_{1j}, \sum_{j=1}^{N_2} X_{2j} \right)$$

and

$$(2.16) \quad (G_1, G_2) \stackrel{st}{=} \left( \sum_{j=1}^{N_1} M_{1j}, \sum_{j=1}^{N_2} M_{2j} \right)$$

have bivariate exponential and bivariate geometric distributions with mean vectors  $(\lambda_1^{-1}, \lambda_2^{-1})$  and  $(p_1^{-1}, p_2^{-1})$ , respectively.

Proof: The proof follows easily by computing the marginal characteristic functions. ||

Finally, we describe how random mixing and random summation are related in the following lemma and corollary. The connection between the two concepts is a key element in the development of the bivariate exponential and geometric AR and ARMA models discussed in Sections 3, 4, and 5.

Lemma 2.17. Let  $(X_{1j}, X_{2j})$  and  $(M_{1j}, M_{2j})$ ,  $j = 1, 2, \dots$ , be as defined in Lemma 2.14. Let  $(N_1, N_2)$  have the bivariate geometric distribution given by (2.3) with  $1 - \pi_1 = p_{10} + p_{11} < 1$ ;  $1 - \pi_2 = p_{01} + p_{11} < 1$ , and be independent of the  $(X_{1j}, X_{2j})$  and  $(M_{1j}, M_{2j})$ . Furthermore, let  $(X_1, X_2)$ ,  $(I_1, I_2)$ ,  $(M_1, M_2)$  and  $(K_1, K_2)$  be as defined in Lemma 2.10. Then  $(Y_1, Y_2) = \left( \sum_{j=1}^{N_1} X_{1j}, \sum_{j=1}^{N_2} X_{2j} \right)$  has the representation  $(I_1 Z_1, I_2 Z_2) + (\pi_1 X_1, \pi_2 X_2)$ , and  $(G_1, G_2) = \left( \sum_{j=1}^{N_1} M_{1j}, \sum_{j=1}^{N_2} M_{2j} \right)$  has the representation  $(I_1 K_1, I_2 K_2) + (M_1, M_2)$ , i.e.

$$(2.18) \quad E[\exp(it_1 Y_1 + it_2 Y_2)] = E[\exp(it_1(I_1 Z_1 + \pi_1 X_1) + it_2(I_2 Z_2 + \pi_2 X_2))]$$

and

$$(2.19) \quad E[\exp(it_1 G_1 + it_2 G_2)] = E[\exp(it_1(I_1 K_1 + M_1) + it_2(I_2 K_2 + M_2))]$$

where  $i = \sqrt{-1}$  and  $-\infty < t_1, t_2 < \infty$ .

Proof: We prove the lemma for the exponential case, the geometric case being similar. Write

$$(\sum_{j=1}^{N_1} X_{1j}, \sum_{j=1}^{N_2} X_{2j}) = (X_{11}, X_{22}) + (\chi(N_1 > 1) \sum_{j=2}^{N_1} X_{1j}, \chi(N_2 > 1) \sum_{j=2}^{N_2} X_{2j})$$

where  $\chi$  denotes the indicator function. Clearly  $(X_{11}, X_{22}) \stackrel{\text{st}}{=} (\pi_1 X_1, \pi_2 X_2)$  and

we are left to show that  $(\chi(N_1 > 1) \sum_{j=2}^{N_1} X_{1j}, \chi(N_2 > 1) \sum_{j=2}^{N_2} X_{2j})$  has the same distribution as  $(I_1 Z_1, I_2 Z_2)$ . Note that

$$(2.20) \quad P(\chi(N_1 > 1) \sum_{j=2}^{N_1} X_{1j} \leq x_1, \chi(N_2 > 1) \sum_{j=2}^{N_2} X_{2j} \leq x_2) \\ = \sum_{i_1=0}^1 \sum_{i_2=0}^1 P(\chi(N_1 > 1) \sum_{j=2}^{N_1} X_{1j} \leq x_1, \chi(N_2 > 1) \sum_{j=2}^{N_2} X_{2j} \leq x_2, \\ \chi(N_1 > 1) = i_1, \chi(N_2 > 1) = i_2).$$

Now, for  $(i_1, i_2) = (1, 1)$  in (2.20) we have

$$(2.21) \quad P(\sum_{j=2}^{N_1} X_{1j} \leq x_1, \sum_{j=2}^{N_2} X_{2j} \leq x_2, N_1 > 1, N_2 > 1)$$

$$= \sum_{n_1=2}^{\infty} \sum_{n_2=2}^{\infty} P\left(\sum_{j=2}^{n_1} X_{1j} \leq x_1, \sum_{j=2}^{n_2} X_{2j} \leq x_2\right) P(N_1=n_1, N_2=n_2).$$

But  $P(N_1=n_1, N_2=n_2) = P(N_1=n_1-1, N_2=n_2-1) P(N_1>1, N_2>1)$ , so that (2.21) is equal to

$$\begin{aligned} & P\left(\sum_{j=1}^{N_1} X_{1j} \leq x_1, \sum_{j=1}^{N_2} X_{2j} \leq x_2\right) P(I_1=1, I_2=1) \\ &= P(Z_1 \leq x_1, Z_2 \leq x_2) P(I_1=1, I_2=1). \end{aligned}$$

For  $(i_1, i_2) = (1, 0)$  in (2.20) we have

$$\begin{aligned} (2.22) \quad & P\left(\sum_{j=2}^{N_1} X_{1j} \leq x_1, 0 \leq x_2, N_1>1, N_2=1\right) \\ &= \sum_{n=2}^{\infty} P\left(\sum_{j=2}^n X_{1j} \leq x_1\right) P(N_1=n, N_2=1). \end{aligned}$$

It is easy to check via (2.3), that  $P(N_1=n, N_2=1) = p_{10}(p_{00}+p_{01})(p_{10}+p_{11})^{n-2} = P(N_1=n-1)P(N_1>1, N_2=1)$ , so that (2.22) is equal to

$$\begin{aligned} & P\left(\sum_{j=1}^{N_1} X_{1j} \leq x_1\right) P(I_1=1, I_2=0) \\ &= P(Z_1 \leq x_1) P(I_1=1, I_2=0). \end{aligned}$$

The case when  $(i_1, i_2) = (0, 1)$  and  $(i_1, i_2) = (0, 0)$  in (2.20) follow similarly.

||

### 3. The General BEAR(m) and BGAR(m) Models

In this section we construct two classes of AR sequences of bivariate random vectors. In each class the sequences are labeled by the parameter  $m$ . We denote the first class of sequences by  $\{\underline{X}(m,n) = (X_1(m,n), X_2(m,n))', n = 0, 1, \dots\}$  and the second class of sequences by  $\{\underline{G}(m,n) = (G_1(m,n), G_2(m,n))', n = 0, 1, \dots\}$ ,  $m = 1, 2, \dots$ . We show that the random vectors  $\underline{X}(m,n)$  and  $\underline{G}(m,n)$  have bivariate exponential and geometric distributions, respectively, with mean vectors that do not depend on  $m$  or  $n$ . Then we discuss the association property for any finite number of random variables belonging to one of the two AR classes. We conclude this section by discussing the autocorrelation structure of the two classes of sequences. Throughout,  $n$  ranges over the nonnegative integers and  $l$  assumes the values 1 or 2. For notation simplicity we suppress the parameter  $m$  since it is fixed throughout the section.

First we construct the class of BEAR(m) sequences. Some notation is needed.

Notation 3.1. Let  $\lambda_1, \lambda_2 \in (0, \infty)$ , let  $\pi_1(n), \pi_2(n) \in (0, 1)$ , and let  $B(n)$  be a  $2 \times 2$  diagonal matrix with  $B(n) = \text{diag}(\pi_1(n), \pi_2(n))$ . Further let  $\underline{E}'(n) = (E_1(n), E_2(n))$  be a sequence of independent bivariate exponential random vectors with mean vector  $(\lambda_1^{-1}, \lambda_2^{-1})'$ , let  $\underline{e}_j$  be a  $m$ -dimensional vector with component  $j$  equal to 1 and the other components equal to 0,  $j = 1, \dots, m$ , and let  $\underline{0}$  denote the  $m$ -dimensional zero vector. Finally let  $\underline{I}'(n) = (I_1(n, 1), \dots, I_1(n, m), I_2(n, 1), \dots, I_2(n, m))$  be a sequence of  $2m$ -dimensional independent random vectors with components assuming values 1 or 0 independent of all  $\underline{E}(n)$ , and let  $A(n, q)$  be a  $2 \times 2$  random diagonal matrix with  $A(n, q) = \text{diag}(I_1(n, q), I_2(n, q))$ ,  $q = 1, \dots, m$ .

We assume that for  $\ell = 1, 2$ ,

$$(3.2) \quad \sum_{j=1}^m P\{(I_{\ell}(n,1), \dots, I_{\ell}(n,m)) = \underline{e}_j'\} = 1 - \pi_{\ell}(n),$$

and that

$$(3.3) \quad P\{(I_{\ell}(n,1), \dots, I_{\ell}(n,m)) = \underline{0}'\} = \pi_{\ell}(n).$$

We define the BEAR(m) sequences as follows.

$$(3.4) \quad \underline{X}(n) = \begin{cases} \underline{E}(n) & n = 0, \dots, m-1 \\ \sum_{q=1}^m A(n,q)\underline{X}(n-q) + B(n)\underline{E}(n) & n = m, m+1, \dots \end{cases}$$

Next we construct the class of BGAR(m) sequences. Some notation is needed.

Notation 3.5. Let  $p_1, p_2, \alpha_1(n), \alpha_2(n) \in (0,1)$  such that  $p_{\ell} \leq \alpha_{\ell}(n)$ , let  $\underline{N}'(n) = (N_1(n), N_2(n))$  be a sequence of independent bivariate geometric random vectors with mean vectors  $(\alpha_{\ell}(n)p_1^{-1}, \alpha_2(n)p_2^{-1})$ , respectively, and let  $\underline{M}'(n) = (M_1(n), M_2(n))$  be a sequence of independent bivariate geometric random vectors with a common mean vector  $(p_1^{-1}, p_2^{-1})$  independent of all  $\underline{N}(n)$ . Further let  $\underline{J}'(n) = (J_1(n,1), \dots, J_1(n,m), J_2(n,1), \dots, J_2(n,m))$  be a sequence of  $2m$ -dimensional independent random vectors with components assuming the values 1 or 0, independent of all  $\underline{N}(n)$  and  $\underline{M}(n)$ . Let  $C(n,q)$  be a  $2 \times 2$  random diagonal matrix with  $C(n,q) = \text{diag}(J_1(n,q), J_2(n,q))$ ,  $q = 1, \dots, m$ .

We assume that for  $\ell = 1, 2$ ,

$$(3.6) \quad \sum_{j=1}^m P\{(J_{\ell}(n,1), \dots, J_{\ell}(n,m)) = \underline{e}_j'\} = 1 - \alpha_{\ell}(n)$$

and that

$$(3.7) \quad P(J_\ell(n,1), \dots, J_\ell(n,m)) = \underline{0}' = \alpha_\ell(n).$$

We define the BGAR(m) sequences as follows.

$$(3.8) \quad \underline{G}(n) = \begin{cases} \underline{M}(n) & n = 0, \dots, m-1 \\ \sum_{q=1}^m C(n,q) \underline{G}(n-q) + \underline{N}(n) & n = m, m+1, \dots \end{cases}$$

Next, we show that  $\underline{X}(n)$  and  $\underline{G}(n)$  have bivariate exponential and geometric distributions, respectively.

Lemma 3.9. For  $n = 0, 1, \dots$ ,  $\underline{X}(n)$  [ $\underline{G}(n)$ ] has a bivariate exponential [geometric] distribution with mean vector  $(\lambda_1^{-1}, \lambda_2^{-1})$  [ $(p_1^{-1}, p_2^{-1})$ ].

Proof: We prove the results of the lemma by an induction argument on  $n$ . For  $n = 0, \dots, m-1$ , the results of the lemma follow by the definition of  $\underline{X}(n)$  and of  $\underline{G}(n)$ . Let us assume that the results of the lemma hold for all nonnegative integers that are less than or equal to  $r$ ,  $r \geq m-1$ , and prove that the results of the lemma hold for  $r+1$ .

Let  $\underline{E}' = (E_1, E_2)$  [ $\underline{M}' = (M_1, M_2)$ ] be a bivariate exponential [geometric] random vector with mean vector  $(\lambda_1^{-1}, \lambda_2^{-1})$  [ $(p_1^{-1}, p_2^{-1})$ ] independent of all  $\underline{E}(n)$  [ $\underline{N}(n)$  and  $\underline{M}(n)$ ]. Then, by the induction assumption, we have for  $\ell = 1, 2$  that

$$X_\ell(r+1) = \begin{cases} E_\ell + \pi_\ell(r+1)E_\ell(r+1), & \text{w.p. } 1 - \pi_\ell(r+1) \\ \pi_\ell(r+1)E_\ell(r+1) & , \text{w.p. } \pi_\ell(r+1) \end{cases}$$

and that

$$G_{\ell}(r+1) \stackrel{\text{st}}{=} \begin{cases} M_{\ell} + N_{\ell}(r+1), & \text{w.p. } 1-\alpha_{\ell}(r+1) \\ N_{\ell}(r+1) & , \text{w.p. } \alpha_{\ell}(r+1), \end{cases}$$

where w.p. stands for 'with probability'. It is easy to check that  $X_{\ell}(r+1)$  [ $G_{\ell}(r+1)$ ] has an exponential [geometric] distribution with mean  $\lambda_{\ell}^{-1}$  [ $p_{\ell}^{-1}$ ]. Consequently the results of the lemma follow. ||

Now we consider the association of any finite collection of the  $X_{\ell}(n)$ 's and of the  $G_{\ell}(n)$ 's.

Lemma 3.10. Let us assume that for  $j = 0, \dots, m-1$ , the random variables  $X_1(j)$ ,  $X_2(j)$  in (3.4) are associated; let  $n_1 < n_2 < \dots < n_r$ , be nonnegative integers and let  $\ell_1, \dots, \ell_r \in \{1, 2\}$ ,  $r = 1, 2, \dots$ . Then the random variables  $X_{\ell_q}(n_q)$ ,  $q = 1, \dots, r$ , are associated.

Proof: Let  $T_{2j} = X_2(j-1)$  and let  $T_{2j-1} = X_1(j-1)$ ,  $j = 1, 2, \dots$ . To prove the result of the lemma it suffices, by Barlow-Proschan (1981, P<sub>1</sub>, p. 30), to show that

(3.11) the random variables  $T_1, \dots, T_r$  are associated for all  $r=1, 2, \dots$ .

We prove (3.11) by an induction argument on  $r$ . For  $r \leq 2m$  (3.11) follows by the lemma assumption and by Barlow-Proschan (1981, P<sub>4</sub>, p. 30). Let us assume that (3.11) holds for  $r$ ,  $r \geq 2m$  and prove that (3.11) holds for  $r+1$ .

By (3.4) the conditional random variable  $T_{r+1} | T_1, \dots, T_r$  is stochastically non-decreasing in  $T_1, \dots, T_r$ . Thus by Barlow-Proschan (1981, Lemma 4.8, p. 147), there is an  $r+1$  argument function  $h$ , nondecreasing in each argument, and a random variable  $U$  independent of  $T_1, \dots, T_r$ , such that  $(T_1, \dots, T_{r+1}) \stackrel{\text{st}}{=} (T_1, \dots, T_r, h(U, T_1, \dots, T_r))$ . By Barlow-Proschan (1981, P<sub>2</sub>, p. 30),  $U$  is

associated and hence by Barlow-Proschan (1981, P<sub>3</sub>, p. 30) the random variables  $U, T_1, \dots, T_r$  are associated. Consequently by Barlow-Proschan (1981, P<sub>3</sub>, p. 30),  $(T_1, \dots, T_{r+1})$  are associated. ||

Using a similar proof we obtain the following.

Lemma 3.12. Let us assume that for  $j = 0, \dots, m-1$ , the random variables  $G_1(j)$ ,  $G_2(j)$  in (3.8) are associated. Let  $n_1 < n_2 < \dots < n_r$ , and  $\ell_1, \dots, \ell_r$ ,  $r = 1, 2, \dots$ , be as in Lemma 3.10. Then the random variables  $G_{\ell_q}(n_q)$ ,  $q = 1, \dots, r$  are associated.

Finally, for each class of sequences, we compute the autocorrelation functions in the case when the marginal processes are stationary.

For the exponential models, put  $\pi_{\ell}(n) = \pi_{\ell}$ , all  $n$ ;  $\ell = 1, 2$ , and let

$$P\{I_{\ell}(n, q) = 1\} = \phi_{\ell}(q), \quad \ell = 1, 2; \quad q = 1, \dots, m$$

such that

$$(i) \phi_{\ell}(q) \geq 0, \text{ and } (ii) \sum_{q=1}^m \phi_{\ell}(q) = 1 - \pi_{\ell}, \quad \ell = 1, 2,$$

as specified in equation (3.2). Define

$$\rho_{X_{\ell}}(k) = \text{Corr}(X_{\ell}(n), X_{\ell}(n+k)), \quad \ell = 1, 2; \quad n = m, m+1, \dots; \quad k = 1, 2, \dots$$

Then

$$(3.13) \quad \rho_{X_{\ell}}(k) = \phi_{\ell}(1)\rho_{X_{\ell}}(k-1) + \phi_{\ell}(2)\rho_{X_{\ell}}(k-2) + \dots + \phi_{\ell}(m)\rho_{X_{\ell}}(k-m)$$

$$\text{with } \text{Var}(X_{\ell}(n)) = \lambda_{\ell}^{-2}, \quad \ell = 1, 2.$$

For the geometric models, put  $\alpha_{\ell}(n) = \alpha_{\ell}$ , all  $n$ ;  $\ell = 1, 2$ , and suppose that

$$P(J_\ell(n, q) = 1) = \gamma_\ell(q), \quad \ell = 1, 2; q = 1, \dots, m$$

such that

$$(i) \gamma_\ell(q) \geq 0, \text{ and } (ii) \sum_{q=1}^m \gamma_\ell(q) = 1 - \alpha_\ell, \quad \ell = 1, 2,$$

as specified in equation (3.6). Define

$$\rho_{G_\ell}(k) = \text{Corr}(G_\ell(n), G_\ell(n+k)), \quad \ell = 1, 2; n = m, m+1, \dots; k = 1, 2, \dots$$

Then

$$(3.14) \quad \rho_{G_\ell}(k) = \gamma_\ell(1)\rho_{G_\ell}(k-1) + \gamma_\ell(2)\rho_{G_\ell}(k-2) + \dots + \gamma_\ell(m)\rho_{G_\ell}(k-m)$$

$$\text{with } \text{Var}(G_\ell(n)) = (1 - p_\ell)p_\ell^{-2}, \quad \ell = 1, 2.$$

Evidently, the marginal correlation structures of the bivariate exponential and geometric sequences, as given in (3.13) and (3.14), respectively, are similar to that of the Gaussian AR(m) process. We note that, in general, even when the marginal processes are stationary, the joint process is not stationary. This is easily seen, for example, by letting  $m=1$  in (3.4) with  $\pi_\ell(n) = \pi_\ell$ , all  $n$ ;  $\ell = 1, 2$ ; choosing  $\underline{E}(n)$  to be an iid sequence of random vectors where  $E_1(n)$  and  $E_2(n)$  are iid exponential random variables for all  $n$ , and letting  $\underline{I}(n)$  be an iid sequence of random vectors for which  $P(I_1(n)=1, I_2(n)=1) = p_{11} \neq (1-\pi_1)(1-\pi_2)$ . A simple computation shows that  $\text{Cov}(X_1(1), X_2(1)) \neq \text{Cov}(X_1(2), X_2(2))$  in this example.

In the next section, we develop models in which the joint processes are also stationary.

#### 4. The Stationary BEAR(1) and BGAR(1) Models

In this section we consider special cases of the BEAR(m) and BGAR(m) models given in Section 3 in which the joint processes are stationary. Throughout this section we put  $m = 1$ , assume that  $\pi_\ell(n)$  and  $\alpha_\ell(n)$  do not vary with  $n$ , and put more structure on the  $\underline{E}(n)$ ,  $\underline{M}(n)$ , and  $\underline{N}(n)$  sequences. We show that for these models, the bivariate distributions of  $\underline{X}(n)$  and of  $\underline{G}(n)$  have a form of the type studied by Arnold (1975). By selecting the  $\underline{E}(n)$ ,  $\underline{M}(n)$ , and  $\underline{N}(n)$  sequences, as defined in Section 3, appropriately, we can obtain well known bivariate distributions. For a stationary BEAR(1) model we obtain the (i) Marshall-Olkin (1967), (ii) Downton (1970), (iii) Hawkes (1972), and (iv) Paulson (1973) bivariate exponential distributions. For a stationary BGAR(1) model we obtain the (i) Esary-Marshall (1973), (ii) Hawkes (1972), and (iii) Paulson-Uppuluri (1972) bivariate geometric distributions. We conclude this section by computing the autocovariance matrices for each model.

First we present the exponential case. Some notation and assumptions are needed.

Let  $m = 1$  and let us assume that  $(I_1(n,1), I_2(n,1))$ , given in (3.1), is an i.i.d. sequence of bivariate random vectors.

For simplicity of notation, denote  $\pi_\ell(n)$  by  $\pi_\ell$ , for  $\ell = 1, 2$ , and let  $p_{ij} = P\{(I_1(n,1), I_2(n,1)) = (i,j)\}$ ,  $i, j = 0, 1$ . Note that by (3.2) and (3.3)

$$(4.1) \quad p_{10} + p_{11} = 1 - \pi_1, \quad p_{01} + p_{11} = 1 - \pi_2.$$

Further let  $(N_1, N_2)$  be a bivariate geometric random vector with parameters  $p_{ij}$ ,  $i, j = 0, 1$  given by (2.3), and let  $\underline{E}(r)$ ,  $r = \pm 1, \pm 2, \dots$ , be an i.i.d. sequence of bivariate exponential random vectors with mean vector  $(\lambda_1^{-1}, \lambda_2^{-1})$

independent of  $(N_1, N_2)$  and all  $(I_1(n,1), I_2(n,1))$ . Note that by Lemma 2.14

$(\sum_{j=1}^{N_1} \pi_1 E_1(-j), \sum_{j=1}^{N_2} \pi_2 E_2(-j))$  is a bivariate exponential random vector with mean vector  $(\lambda_1^{-1}, \lambda_2^{-1})$ . We assume that

$$(4.2) \quad \underline{E}(0) = \left( \sum_{j=1}^{N_1} \pi_1 E_1(-j), \sum_{j=1}^{N_2} \pi_2 E_2(-j) \right)'$$

Define  $A(n)$  to be the  $2 \times 2$  diagonal random matrix  $A(n) = \text{diag}(I_1(n,1), I_2(n,1))$  and  $B$  to be the  $2 \times 2$  diagonal matrix  $B = \text{diag}(\pi_1, \pi_2)$ . The stationary BEAR(1) model is defined as follows.

$$(4.3) \quad \underline{X}(n) = \begin{cases} \underline{E}(0) & n=0 \\ A(n)\underline{X}(n-1) + B\underline{E}(n) & n=1, 2, \dots \end{cases}$$

We now state and prove a characterization of  $\underline{X}(n)$ .

Lemma 4.4 Let  $\underline{X}(n)$  be defined by (4.3). Then for  $n = 0, 1, \dots$ ,

$$\underline{X}(n) \stackrel{\text{st}}{=} \underline{E}(0)$$

where  $\underline{E}(0)$  is given in (4.2).

Proof: We prove the result of the lemma by an induction argument on  $n$ . By definition the result of the lemma holds for  $n = 0$ . Let us assume that the result of the lemma holds for  $n, n \geq 0$ .

Note that

$$\underline{E}'(0) = \left( \sum_{j=1}^{N_1} \pi_1 E_1(-j), \sum_{j=1}^{N_2} \pi_2 E_2(-j) \right) =$$

$$(\chi(N_1 > 1) \pi_1 \sum_{j=2}^{N_1} E_1(-j), \chi(N_2 > 1) \pi_2 \sum_{j=2}^{N_2} E_2(-j))$$

$$+ (\pi_1 E_1(-1), \pi_2 E_2(-1)),$$

where  $\chi(\cdot)$  denotes the indicator function, and that the two summands are independent random vectors.

Now by Lemma 2.17, (4.1) and the induction assumption

$$(\chi(N_1 > 1) \pi_1 \sum_{j=2}^{N_1} E_1(-j), \chi(N_2 > 1) \pi_2 \sum_{j=2}^{N_2} E_2(-j))$$

$$\stackrel{st}{=} (I_1(n,1) \pi_1 \sum_{j=1}^{N_1} E_1(-j), I_2(n,1) \pi_2 \sum_{j=1}^{N_2} E_2(-j))$$

$$\stackrel{st}{=} (I_1(n,1) X_1(n), I_2(n,1) X_2(n)).$$

Further by the definition of  $\underline{E}(r)$

$$(\pi_1 E_1(-1), \pi_2 E_2(-1)) \stackrel{st}{=} (\pi_1 E_1(n), \pi_2 E_2(n)).$$

Since the random vectors  $(\pi_1 E_1(n), \pi_2 E_2(n))$  and  $(I_1(n,1) X_1(n), I_2(n,1) X_2(n))$  are independent we have by (3.4) that

$$(\pi_1 \sum_{j=1}^{N_1} E_1(-j), \pi_2 \sum_{j=1}^{N_2} E_2(-j)) \stackrel{st}{=} (\pi_1 E_1(n), \pi_2 E_2(n))$$

$$+ (I_1(n,1) X_1(n), I_2(n,1) X_2(n)) = \underline{X}'(n+1).$$

Consequently the result of the lemma follows. ||

In the following remark we show that many interesting bivariate distributions are possible in Lemma 4.4. See Block (1977) for further details.

Remark 4.5. (a) Let  $E$  be an exponential random variable with mean  $\theta$ ,  $0 < \theta < (\lambda_1 + \lambda_2)^{-1}$ , let  $\pi_1 = \lambda_1 \theta$ ,  $\pi_2 = \lambda_2 \theta$  and let  $\underline{E}(1) = (\pi_1^{-1} E, \pi_2^{-1} E)$ :

(i) If  $p_{00} = 0$ ,  $p_{01} = \pi_1$ ,  $p_{10} = \pi_2$  and  $p_{11} = 1 - (\lambda_1 + \lambda_2)\theta$ , then the resulting  $\underline{X}(n)$  has independent components.

(ii) Let  $b_1, b_2, b_3$  be nonnegative real numbers such that  $\lambda_1 = b_1 + b_{12}$  and  $\lambda_2 = b_2 + b_{12}$ . If  $p_{00} = \theta b_{12}$ ,  $p_{10} = \theta b_2$ ,  $p_{01} = \theta b_1$  and  $p_{11} = 1 - \theta(b_1 + b_2 + b_{12})$  then the resulting  $\underline{X}(n)$  has a Marshall-Olkin (1967) bivariate exponential distribution.

(b) Let  $\underline{E}(1)$  have independent components:

(i) If  $p_{11} = \gamma(1+\gamma)^{-1}$ ,  $0 < \gamma$ ,  $p_{10} = p_{01} = 0$ , and  $p_{00} = (1+\gamma)^{-1}$  then the resulting  $\underline{X}(n)$  has the Downton (1970) bivariate exponential distribution.

(ii) With different requirements on  $p_{ij}$   $i, j = 0, 1$ , the resulting  $\underline{X}(n)$  has the Hawkes (1972) and the Paulson (1973) bivariate exponential distribution. See Block (1977) for details.

Next we present the geometric case. Some notation and assumptions are needed.

Let  $m = 1$  and let us assume that  $(J_1(n, 1), J_2(n, 1))$ , given in (3.5), is an i.i.d. sequence of bivariate random vectors.

For simplicity of notation denote  $\alpha_\ell(n)$  by  $\alpha_\ell$ ,  $\ell = 1, 2$ , and let  $p_{ij} = P\{(J_1(n, 1), J_2(n, 1)) = (i, j)\}$ ,  $i, j = 0, 1$ . Note that by (3.6) and (3.7)

$$(4.6) \quad p_{10} + p_{11} = 1 - \alpha_1, \quad p_{01} + p_{11} = 1 - \alpha_2.$$

Further let  $(Q_1, Q_2)$  be a bivariate geometric random vector with parameters

$p_{ij}$ ,  $i, j = 0, 1$ , given by (2.3). Let  $p_1, p_2 \in (0, 1)$  such that  $p_\ell \leq \alpha_\ell$ , and let  $\underline{N}(r)$ ,  $r = \pm 1, \pm 2, \dots$ , be an i.i.d. sequence of bivariate geometric random vectors with mean vector  $(\alpha_1 p_1^{-1}, \alpha_2 p_2^{-1})$  independent of  $(Q_1, Q_2)$  and all  $(J_1(n, 1), J_2(n, 1))$ . Note that by Lemma 2.14  $(\sum_{j=1}^{Q_1} N_1(-j), \sum_{j=1}^{Q_2} N_2(-j))$  is a bivariate geometric random vector with mean vector  $(p_1^{-1}, p_2^{-1})$ . We assume that

$$(4.7) \quad \underline{M}(0) = \left( \sum_{j=1}^{Q_1} N_1(-j), \sum_{j=1}^{Q_2} N_2(-j) \right)'.$$

Define  $C(n)$  to be the  $2 \times 2$  diagonal random matrix  $C(n) = \text{diag}(J_1(n, 1), J_2(n, 1))$ .

The stationary BGAR(1) model is defined as follows.

$$(4.8) \quad \underline{G}(n) = \begin{cases} \underline{M}(0) & n=0 \\ C(n)\underline{G}(n-1) + \underline{N}(n) & n=1, 2, \dots \end{cases}$$

We now state and prove a characterization of  $\underline{G}(n)$ .

Lemma 4.9. Let  $\underline{G}(n)$  be defined by (4.8). Then for  $n = 0, 1, \dots$ ,

$$\underline{G}(n) \stackrel{\text{st}}{=} \underline{M}(0)$$

where  $\underline{M}(0)$  is given in (4.7).

Proof: We prove the result of this lemma by using a similar argument to the one used in the proof of Lemma 4.4. ||

In the following remark we show that many interesting bivariate distributions are possible in Lemma 4.9. See Block (1977) for further details.

Remark 4.10. (a) Let  $\alpha_1 = p_1$ ,  $\alpha_2 = p_2$ , thus  $\underline{N}(1) = (1, 1)$ :

(i) Let  $\theta_1, \theta_2, \theta_{12} \in (0, 1)$  and let  $1 - \alpha_1 = \theta_1 \theta_{12}$ ,  $1 - \alpha_2 = \theta_2 \theta_{12}$ . Further let

$p_{11} = \theta_1 \theta_2 \theta_{12}$ ,  $p_{10} = \theta_1 (1-\theta_2) \theta_{12}$ ,  $p_{01} = (1-\theta_1) \theta_2 \theta_{12}$  and  $p_{00} = 1 - p_{11} - p_{01} - p_{10}$ .

Then the resulting  $\underline{G}(n)$  has the Esary-Marshall (1973) bivariate geometric distribution in the narrow sense.

(ii) If  $p_{11} = (1-\alpha_1)(1-\alpha_2)$ ,  $p_{01} = \alpha_2(1-\alpha_2)$ ,  $p_{10} = (1-\alpha_1)\alpha_2$  and  $p_{00} = \alpha_1\alpha_2$  then the resulting  $\underline{G}(n)$  has independent components.

(iii) With no special requirements on the  $p_{ij}$   $i, j = 0, 1$ , the resulting  $\underline{G}(n)$  has the Esary-Marshall (1973) bivariate geometric distribution in the wide sense, or equivalently, the bivariate geometric distribution due to Hawkes (1972).

(b) If  $\underline{N}(1)$  has independent components, the resulting  $\underline{G}(n)$  has the Paulson and Uppuluri (1972) distribution.

Finally, we give the autocovariance matrices for the stationary BEAR(1) and BGAR(1) models. Let  $\underline{\Sigma}_X = \text{Var}\{\underline{X}(n)\}$  be the variance-covariance matrix of  $\underline{X}(n)$ , and  $\underline{\Sigma}_G = \text{Var}\{\underline{G}(n)\}$  be the variance-covariance matrix of  $\underline{G}(n)$ . Note that  $\underline{\Sigma}_X$  and  $\underline{\Sigma}_G$  are independent of  $n$  by Lemmas 4.4 and 4.9, respectively. Define  $\Gamma_X(k) = \text{Cov}\{\underline{X}(n+k), \underline{X}(n)\}$ ,  $k = 0, 1, 2, \dots$ , and  $\Gamma_G(k) = \text{Cov}\{\underline{G}(n+k), \underline{G}(n)\}$ ,  $k = 0, 1, 2, \dots$ , and note that  $\Gamma_X(0) = \underline{\Sigma}_X$  and  $\Gamma_G(0) = \underline{\Sigma}_G$ . In view of (4.3) and (4.8), it is easy to see that  $\Gamma_X(k) = A\Gamma_X(k-1)$  and  $\Gamma_G(k) = C\Gamma_G(k-1)$ ,  $k = 1, 2, \dots$ , respectively, where  $A$  and  $C$  are the  $2 \times 2$  diagonal matrices defined by  $A = \text{diag}\{1-\pi_1, 1-\pi_2\}$  and  $C = \text{diag}\{1-\alpha_1, 1-\alpha_2\}$ . Hence, for the stationary BEAR(1) model we have

$$(4.11) \quad \Gamma_X(k) = A^k \underline{\Sigma}_X; \quad \Gamma_X(-k) = \Gamma_X'(k), \quad k=0, 1, 2, \dots,$$

and for the stationary BGAR(1) model we have

$$(4.12) \quad \Gamma_G(k) = C^k \underline{\Sigma}_G; \quad \Gamma_G(-k) = \Gamma_G'(k), \quad k=0, 1, 2, \dots$$

### 5. The BEARMA( $m_1, m_2$ ) and BGARMA( $m_1, m_2$ ) Models

Using the results of Section 3 and the results of Langberg-Stoffer (1985) for MA sequences, we construct four classes of ARMA sequences of bivariate random vectors. In each class the sequences are labeled by the parameters  $m_1, m_2$ . We denote the first two classes of sequences by  $\{\underline{Z}'(j, m_1, m_2, n) = (Z_1(j, m_1, m_2, n), Z_2(j, m_1, m_2, n)), n = 0, 1, \dots\}$ ,  $j = 1, 2$ , and the other two classes of sequences by  $\{\underline{L}'(j, m_1, m_2, n) = (L_1(j, m_1, m_2, n), L_2(j, m_1, m_2, n)), n = 0, 1, \dots\}$ ,  $j = 1, 2$ . We show that the random vector  $\underline{Z}'(j, m_1, m_2, n)$  [ $\underline{L}'(j, m_1, m_2, n)$ ] has a bivariate exponential [geometric] distribution with a mean vector that does not depend on  $j, m_1, m_2$ , or  $n$ . Then we discuss the association property of any finite number of random variables belonging to one of the four ARMA classes. For notational simplicity we suppress the parameters  $m_1$  and  $m_2$  since they are fixed throughout this section.

First we construct the two classes of BEARMA( $m_1, m_2$ ) sequences. Some notation is needed.

Notation 5.1. Let  $\underline{X}(n)$  be a BEAR( $m_1$ ) sequence given by (3.4), and let  $\underline{Y}(n)$  be an  $m_2$ -dependent BEMA sequence as given by Langberg-Stoffer (1985, p. 7), independent of the  $\underline{X}(n)$  sequence. Further let  $\underline{V}'(n) = (V_1(n), V_2(n))$  be a sequence of independent bivariate random vectors with components assuming the values 1 or 0, independent of the  $\underline{X}(n)$  and  $\underline{Y}(n)$  sequences and let  $P\{V_\ell(n) = 1\} = \pi_\ell(n)$ ,  $0 < \pi_\ell(n) < 1$ ,  $\ell = 1, 2$ .

We define the two BEARMA( $m_1, m_2$ ) sequences as follows.

$$(5.2) \quad (Z_1(1, n), Z_2(1, n)) = ((1 - \pi_1(n))Y_1(n), (1 - \pi_2(n))Y_2(n)) \\ + (V_1(n)X_1(n), V_2(n)X_2(n))$$

$$(5.3) \quad (Z_1(2,n), Z_2(2,n)) = ((1-\pi_2(n))X_2(n), (1-\pi_2(n))X_2(n)) \\ + (V_1(n)Y_1(n), V_2(n)Y_2(n)).$$

Now we construct the two classes of BGARMA( $m_1, m_2$ ) sequences. Some notation is needed.

Notation 5.4. Let  $\delta_1, \delta_2, \beta_1, \beta_2 \in (0,1)$  such that  $\delta_\ell \leq \beta_\ell$ ,  $\ell = 1,2$ , and let  $\underline{G}(1,n)$  [ $\underline{G}(2,n)$ ] be a BGAR( $m_1$ ) sequence with mean vector  $(\beta_1, \delta_1^{-1}, \beta_2 \delta_2^{-1})$  [ $(\delta_1^{-1}, \delta_2^{-1})$ ] given by (3.8). Further let  $\underline{H}(1,n)$  [ $\underline{H}(2,n)$ ] be an  $m_2$ -dependent BGMA sequence with mean vector  $(\delta_1^{-1}, \delta_2^{-1})$  [ $(\beta_1 \delta_1^{-1}, \beta_2 \delta_2^{-1})$ ] given by Langberg-Stoffer (1985, p. 8), independent of all  $\underline{G}(1,n)$  [ $\underline{G}(2,n)$ ]. Finally let  $\underline{U}(n) = (U_1(n), U_2(n))$  be an i.i.d. sequence of bivariate random vectors with components assuming the values 1 or 0 independent of all the previous random vectors such that  $P(U_\ell(n) = 1) = 1 - \beta_\ell$ ,  $\ell = 1,2$ .

We define the two BGARMA( $m_1, m_2$ ) sequences as follows.

$$(5.5) \quad (L_1(1,n), L_2(1,n)) = (G_1(1,n), G_2(1,n)) \\ + (U_1(n)H_1(1,n), U_2(n)H_2(1,n)),$$

$$(5.6) \quad (L_1(2,n), L_2(2,n)) = (H_1(2,n), H_2(2,n)) \\ + (U_1(n)G_1(2,n), U_2(n)G_2(2,n)).$$

Next we show that  $\underline{Z}(j,n)$  and  $\underline{L}(j,n)$  have bivariate exponential and geometric distributions, respectively.

Lemma 5.7. For  $j = 1,2$ , and  $n = 0,1,\dots$ ,  $\underline{Z}(j,n)$  [ $\underline{L}(j,n)$ ] has a bivariate exponential [geometric] distribution with mean vector  $(\lambda_1^{-1}, \lambda_2^{-1})$  [ $(\delta_1^{-1}, \delta_2^{-1})$ ].

Proof: By Lemma 3.9  $\underline{X}(n) [\underline{G}(j,n)]$  has a bivariate exponential [geometric] distribution. By Langberg-Stoffer (1985, Corollary 3.8),  $\underline{Y}(n) [\underline{H}(j,n)]$  has a bivariate exponential [geometric] distribution. Consequently the result of the lemma follow by the four definitions and by Lemma 2.10. ||

Now we consider the association property of any finite number of random variables belonging to one of the four ARMA classes. We assume that the assumptions of Langberg-Stoffer (1985), Lemmas 3.12 and 3.13) are satisfied. We need the following lemma.

Lemma 5.8. Let  $S_1, \dots, S_r, T_1, \dots, T_r$  be nonnegative random variables. Let us assume that  $S_1, \dots, S_r$  and  $T_1, \dots, T_r$  are associated, and that the random vectors  $(S_1, \dots, S_r)$  and  $(T_1, \dots, T_r)$  are independent. Then the random variables  $S_1 T_1, \dots, S_r T_r$  are associated.

Proof: Let  $\underline{T} = (T_1, \dots, T_r)$ , let  $\underline{W} = (S_1 T_1, \dots, S_r T_r)$ , and let  $f, g$  be two nonnegative functions each with  $r$  arguments, nondecreasing in each argument.

The components of the conditional random vector  $\underline{W}|\underline{T}$  are nondecreasing functions of the associated random variables  $S_1, \dots, S_r$ . Thus, by Barlow-Proschan (1981,  $P_3$ , p. 30), the components of  $\underline{W}|\underline{T}$  are associated. Hence,

$$E(\text{cov}(f(\underline{W}), g(\underline{W}))|\underline{T}) \geq 0.$$

$E(f(\underline{W})|\underline{T})$  and  $E(g(\underline{W})|\underline{T})$  are two nondecreasing functions of the associated random variables  $T_1, \dots, T_r$ . Thus, by Barlow-Proschan (1981,  $P_3$ , p. 30), the two random variables  $E(f(\underline{W})|\underline{T})$  and  $E(g(\underline{W})|\underline{T})$  are associated. Hence

$$\text{cov}(E(f(\underline{W})|\underline{T}), E(g(\underline{W})|\underline{T})) \geq 0.$$

Note that

$$\begin{aligned} \text{cov}(f(\underline{W}), g(\underline{W})) &= E(\text{cov}(f(\underline{W}), g(\underline{W})) | \underline{I}) \\ &+ \text{cov}(E(f(\underline{W}) | \underline{I}), E(g(\underline{W}) | \underline{I})). \end{aligned}$$

Consequently the result of the lemma follows. ||

Lemma 5.9. Let us assume that for  $n = 0, 1, \dots$ ,  $V_1(n)$  and  $V_2(n)$  are associated. Let  $n_1 < n_2 < \dots < n_r$ , and  $\ell_1, \dots, \ell_r$ ,  $r = 1, 2, \dots$ , be as in Lemma 3.10. Then the  $Z_{\ell_q}(1, n_q)$  [ $Z_{\ell_q}(2, n_q)$ ],  $q = 1, \dots, r$  are associated.

Proof: By Lemma 3.10,  $X_{\ell_q}(n_q)$ ,  $q = 1, \dots, r$  are associated. By Langberg-Stoffer (1985, Lemma 3.12), the random variables  $Y_{\ell_q}(n_q)$ ,  $q = 1, \dots, r$  are associated. By our assumption and by Barlow-Proschan (1981,  $P_4$ , p. 30),  $V_{\ell_q}(n_q)$ ,  $q = 1, \dots, r$  are associated.

Thus, by Lemma 4.8,  $V_{\ell_q}(n_q) X_{\ell_q}(n_q)$ ,  $q = 1, \dots, r$ , and  $V_{\ell_q}(n_q) Y_{\ell_q}(n_q)$ ,  $q = 1, \dots, r$ , are associated. Now, Barlow-Proschan (1981,  $P_4$ , p. 30), the random variables  $\{X_{\ell_q}(n_q), V_{\ell_q}(n_q) Y_{\ell_q}(n_q), q=1, \dots, r\}$  and the random variables  $\{Y_{\ell_q}(n_q), V_{\ell_q}(n_q) X_{\ell_q}(n_q), q=1, \dots, r\}$  are associated. Consequently the results of the lemma follow by (5.2), (5.3) and by Barlow-Proschan (1981,  $P_3$ , p. 30). ||

Using a similar argument we obtain the following.

Lemma 5.10. Let us assume that  $U_1(0)$ ,  $U_2(0)$  are associated; let  $n_1 < n_2 < \dots < n_r$ , and  $\ell_1, \dots, \ell_r$ ,  $r = 1, 2, \dots$ , be as in Lemma 3.10. Then the  $L_{\ell_q}(1, n_q)$  [ $L_{\ell_q}(2, n_q)$ ],  $q = 1, \dots, r$ , are associated.

Langberg-Stoffer (1985, Section 4) present inequalities and probability

bounds for the bivariate point processes related to the bivariate exponential or geometric MA sequences. We note that all the results given by Langberg-Stoffer (1985, Section 4) hold for the bivariate point processes related to the bivariate exponential or geometric AR sequences, given in Sections 3 and 4, and to the ARMA sequences given in this section, provided that they are associated.

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